The Virtues of Multicontinuum Mechanics for Composites Analysis

DOI: 10.2514/6.2009-2477

CITATION
1

READS
55

4 authors, including:

Emmett Nelson
E Nelson Engineering, LLC
18 PUBLICATIONS 84 CITATIONS
SEE PROFILE

Andrew Christian Hansen
University of Wyoming
32 PUBLICATIONS 745 CITATIONS
SEE PROFILE

Ray S Fertig III
University of Wyoming
59 PUBLICATIONS 285 CITATIONS
SEE PROFILE

Some of the authors of this publication are also working on these related projects:

Aeroelastic Optimization for Improved Durability View project

Thermo-mechanical modeling of chopped fiber composites View project
Accurate failure predictions of composite structures are often difficult to obtain given the diverse challenges encountered by the analyst. Meaningful analysis for composite structures must strike a balance between modeling damage occurring at the microscale and determining its effect at the macroscale. These divergent goals must also be accomplished in an efficient scheme that delivers significant results for the analyst.

Efforts at Firehole Technologies toward predicting composite mechanics focus on efficiently crossing multiple geometric scales to use microstructural information for the prediction of composite material behavior. To begin, a finite element-based analysis is utilized to bring stress/strain resolution from the laminate level down to the individual lamina. Embedded in the lamina analysis is a multicontinuum theory algorithm that further decomposes lamina stress and strain fields down to fiber and matrix constituent level stress/strain fields.

The idea of a multicontinuum is a new school of thought when considering the analysis of composites. The article presented here will focus on the virtues of multicontinuum mechanics as compared to the two common methods of analyzing composite lamina: continuum mechanics based methods, and micromechanics based methods.

Nomenclature

- $\mathbf{e}$ Composite strain tensor.
- $\mathbf{e}_\beta$ Constituent $\beta$ strain tensor ($\beta = f$ (fiber), $m$ (matrix)).
- $\sigma$ Composite stress tensor.
- $\sigma_\beta$ Constituent $\beta$ stress tensor ($\beta = f$ (fiber), $m$ (matrix)).
- $D$ Volume region of integration.
- $V$ Volume.
- $\sigma_{ij}$ Composite stresses referenced to the local lamina ($i,j = 1$ to 3) or global laminate ($i,j = x$ to $z$) coordinate system.
- $\sigma_{ij\beta}$ Constituent $\beta$ stresses referenced to the local lamina coordinate system ($\beta = f$ (fiber), $m$ (matrix)); ($i,j = 1$ to 3).
- $\phi_\beta$ Constituent $\beta$ volume fraction ($\beta = f$ (fiber), $m$ (matrix)).
- $\{a\}$ Vector relating constituent to composite thermal strains.
- $[A]$ 6 x 6 matrix relating constituent to composite mechanical strains.

---

1 Chief Operations Officer, AIAA member
2 Vice President of Business Development, AIAA member
3 Applied Research Engineer
4 Professor of Mechanical Engineering, Associate Dean
1. Introduction

Virtually every aspect of a finite element structural analysis includes an assumption. Structural analysis is the collision of opposing forces. One must deal with accurately representing details of the structure; at the same a solution that is computationally feasible, easily understood is required in a timely manner. Appropriate assumptions are the foundation of structural analysis and make spanning multiple geometric scales possible.

Accurately approximating the discrepancy in geometric scale in a composite structure is daunting from a structural analyst’s point of view. A single carbon fiber has a diameter of 5.2 x10^{-6} meters [1], the thickness of a single lamina is commonly 1.27 x10^{-4} meters, yet a structure can easily exceed length scales of multiple meters. The interaction between the fiber and matrix constituents is important in capturing the response of a structure [2]; however it is impossible to capture individual fibers in a finite element model of a structure.

A standard, widely recognized assumption for analysis of composites has yet to be realized. Common practice in the aerospace industry is to homogenize the composite lamina or laminate into a single orthotropic material, thus ignoring any interactions between constituents. Many researchers have recognized the need for microscale stress or strain information in order to accurately capture the composite response and proposed multiscale solutions [3], [4], [5]. These methods use micromechanics models coupled with structural analysis in order to gain the needed insight into constituent behavior. However these methods have yet to gain traction in industry.

2. Multicontinuum Mechanics

To establish a framework where microstructural information about a composite can be efficiently captured in structural analysis, the idea of the multicontinuum mechanics was first proposed by Garnich and Hansen [6]. The fundamental principle defining the multicontinuum idea is to strike a balance between completely homogenizing the composite lamina and burdening computational resources by calling a detailed micromechanics model at every integration point within a finite element model.

The concept of a multicontinuum is based on the composite to constituent stress and strain decomposition first described by Hill [7]. The originally proposed Garnich and Hansen [6] method was later adapted for failure analysis by Mayes and Hansen [8],[9]. More recently, Nelson and Hansen [10] developed a modified von Mises failure criterion and provided improved solutions.

The Multicontinuum Theory (MCT) is an extension of the continuum mechanics to multiple materials. It extends the fundamental premise of continuum mechanics: that all physical quantities of interest represent average values of the quantity, only in the multicontinuum, the averaging occurs over two material volumes whose physical
dimensions are small compared to the physical dimensions of the system of interest, yet large enough to capture the average behavior. A schematic depiction of the multicontinuum hypothesis is shown in Figure 1.

\[ \sigma_\alpha = \frac{1}{V_\alpha} \int_{D_\alpha} \sigma_\alpha (x) \, dV_\alpha \]

\[ \sigma_\beta = \frac{1}{V_\beta} \int_{D_\beta} \sigma_\beta (x) \, dV_\beta \]

**Figure 1. Schematic Depiction of the Multicontinuum Hypothesis**

The foundation of the multicontinuum is to build upon traditional continuum mechanics and utilize micromechanics where proven most accurate. Traditional continuum mechanics are used to determine the stress and strain fields for a composite at a point of interest. Then using predetermined relationships between the composite and constituents, average constituent stresses and strains are decomposed from the composite results. Point-wise stress evaluation in the microstructure is not possible or desirable using this approach. Micromechanics analysis is used to establish the needed relationships between the composite and constituents. A detailed derivation of Multicontinuum Theory is provided in Appendix A. The remainder of the article will present the fundamentals of the multicontinuum concept allowing the reader to understand the value the MCT balance provides.

### 3. Analysis of Metals

To develop the idea of a multicontinuum and its application to composite materials, first consider metals analysis—a very mature field compared to composites. Traditionally, structural analysis of metals was conducted using continuum mechanics on an assumed homogeneous material. This approach has a rich heritage of success and can easily be applied to structural analysis using the finite element method. More recently, micromechanics analysis of metals has advanced, which treats the metal as a heterogeneous material. Both approaches have been successful in appropriate situations. In the following section we will discuss the value of each approach.

#### 3.1 Metals as homogeneous materials

The fundamental premise underlying continuum mechanics is that all physical quantities of interest represent average values of the quantity where the averaging occurs over a material volume whose physical dimensions are small compared to the physical dimensions of the system of interest, yet large enough to capture the average behavior. A schematic depiction of the continuum hypothesis is shown in Figure 2.

\[ \sigma = \frac{1}{V} \int_D \sigma (x) \, dV \]

**Figure 2. Schematic Depiction of the Continuum Hypothesis**

In Figure 1, \( \sigma \) is the stress field of the steel point shown in the micrograph. Although local stress evaluation throughout the continuum point would reveal large deviations, it is the average stress over the point that has proven...
most successful for structural analysis purposes. Continuum mechanics has been so successful the continuum average stress is now commonly referred to as the stress in the material without debate. The approach has been applied to analysis of metals providing predictions for complicated problems involving many loading schemes such as monotonic failure, cyclic loading, creep, plastic and viscoelastic behavior.

3.2 Micromechanics
Micromechanics, as applied to metals, have seen an increase in model sophistication with the advances made in finite element analysis (FEA) computing, yet accurate failure predictions using peak stress evaluation from micromechanics has yet to be achieved. The difficulty in peak stress evaluation using micromechanics is accurately representing the details of the microstructure within a reasonably sized volume. Because of this, the representative volume element (RVE) was developed, which is the smallest repeating volume that represents the entire volume of interest. In steels, researchers have studied the local stress distribution within various RVE models producing the following noteworthy results: micromechanics modeling research of steels is quite successful in producing both effective properties. However, examination of local stress variation through the micromechanical model produced large scatter in stress maxima that is highly dependent on the RVE [11], [12]. Even though metals commonly have a highly granular microstructure, traditional continuum mechanics provides undeniably accurate solutions for structural analysis.

4. Analysis of Composites
A composite material by nature contains more than one material; for our discussion consider a unidirectional composite with fiber and matrix constituents. The presence of two materials uniquely distinguishes a composite from a metal. The geometric scale of each material in a composite is commonly an order of magnitude larger than the grains in a metal. Next consider the same two types of analysis methods, composite homogenization and micromechanics, with respect to the analysis of composites.

4.1 Composite homogenization
As noted above, the continuum hypothesis has been used successfully as a basis for structural analysis for nearly a century. However, when applied to composite materials as a single homogeneous continuum, the theory has provided only limited success with regards to structural analysis and failure predictions [13]. This limited success is primarily attributed to the fact the interactions between constituents are neglected in the traditional continuum approach.

Interactions between constituents create large internal stresses that are not captured when the composite is assumed to be a homogenous material; as in the continuum hypothesis. To highlight these internal stresses the following example is presented. Consider a biaxial compression test of a glass/epoxy unidirectional composite material with a fiber volume fraction of 60 percent where \(\sigma_{22} = \sigma_{33} = -10 \text{ MPa}\), as indicated schematically in Figure 3.

\[
\sigma_{33} = -10 \text{ MPa} \\
\sigma_{22} = -10 \text{ MPa}
\]

**Figure 3. Biaxially loading of a Unidirectional Composite**

For this biaxial stress state, it can be shown that the corresponding volume-averaged fiber stress tensor is given by:
Note that the fiber stress state is fully three-dimensional, and furthermore, the stress in the fibers in the unloaded direction is 43 percent of the applied composite biaxial stress. These “hidden” constituent stresses may contribute substantially to failure, yet they do not appear in the composite stress field because they self-equilibrate with the matrix stresses.

Next consider a unidirectional composite subjected to unconstrained cooling from 21°C to -196°C. The stress state in the lamina is zero, as shown in Equation 2, due to the fact that the lamina is unconstrained. The fiber stress for this lamina is shown in Equation 3 and the matrix stress is shown in Equation 4.

\[
\begin{bmatrix}
4.29 & 0 & 0 \\
0 & -10.8 & 0 \\
0 & 0 & -10.8
\end{bmatrix}\text{ MPa}.
\tag{1}
\]

Note that the constituent stress states are again fully three-dimensional. For this particular material, unconstrained cooling is sufficient to cause the matrix of the lamina to crack. Yet by homogenizing the composite into a single continuum, the stress field is exactly zero.

These examples highlight the importance of capturing constituent behavior under both mechanical and thermal loading, and show that models that simply homogenize composite behavior cannot accurately capture relevant material behavior in a composite.

4.2 Micromechanics

To overcome the limitations of considering the composite as a single continuum material, much research has been devoted to micromechanics of composites. A great deal of success has been achieved in predicting effective composite properties and studying the effects of microstructural parameters on macrostructural behavior [14],[15]. But similar to metals, limited success has been achieved when predicting composite failure by evaluating peak stress within a micromechanics model [5].

Now, let us discuss why micromechanics models have found only limited success when evaluating peak stress values for failure prediction. Figure 4(a) shows a micrograph of a unidirectional composite. Notice the highly random nature of the composite. The development of a detailed micromechanics model to map the exact microstructure in the point may be possible, but it is generally accepted that this approach is impractical for analysis due to the large variation in geometric scale.

Rather than model the entire composite volume, researchers have developed a variety of RVEs that attempt to capture the salient microstructural features of the composite using a reasonably sized volume. Two examples of
these RVEs are shown in Fig. 4b-c. Figure 4b depicts an RVE used in a finite element micromechanics model based on an idealized microstructure (hexagonal fiber packing). Figure 4c illustrates an RVE for a random microstructure.

![Figure 4b](image1.jpg)  
![Figure 4c](image2.jpg)

Figure 4. Unidirectional composite microstructure and finite element micromechanics models.

We investigated the effect that a particular RVE has on the predicted composite stress using finite element micromechanics models of a carbon/epoxy unidirectional lamina loaded in transverse tension. Three different RVE were used to model the lamina: one with hexagonal fiber packing and two with different random fiber packings. The constituent properties and fiber volume fractions are the same in each model. Each RVE was subjected to the same transverse tensile strain; the average and peak stresses found in the various models are shown in Table 1.

<table>
<thead>
<tr>
<th></th>
<th>Hexagonal Packing</th>
<th>Random Packing #1</th>
<th>Random Packing #2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Average Transverse Tensile Stress</strong></td>
<td>10.0 MPa</td>
<td>10.0 MPa</td>
<td>10.0 MPa</td>
</tr>
<tr>
<td><strong>Peak Transverse Tensile Stress</strong></td>
<td>12.1 MPa</td>
<td>25.6 MPa</td>
<td>17.3 MPa</td>
</tr>
</tbody>
</table>

Study of Table 1 shows that the average stress is independent of the representative volume element. However, the peak stress varies by more than a factor of two for the exact same loading, thus the peak stress given is highly dependent on the representative volume element. Similar to metals, these given peak stresses may be accurate for the RVE. It is the idealization of the microstructure that causes failure predictions made from evaluating peak stresses to not compare well with experimental data.

Another hindrance on the applicability of micromechanics to structural analysis of composite is the difficulty in coupling micromechanics to structural FEA. Several methods have been developed which couple a micromechanics model to integration points within structural analysis, but most methods have proven computationally inefficient and require a large amount of user expertise [16].

Micromechanics models have proven quite valuable in the investigation of constitutive behavior and in predicting the effect of varying microstructural parameters have on macro-structural behavior. Similar to their use in modeling metals, micromechanical models of composites provide very accurate predictions of effective properties and averaged stress fields. However, the computational expense and the fact that peak stress predictions are highly dependent on the representative volume element continue to impede the effective use of micromechanics in structural analyses.

**5. Advantages of Multicontinuum Theory**

Recall that micromechanics modeling has consistently proven successful in predicting effective material properties. Average stress prediction is a direct function of the effective properties. Therefore, by evaluating the micromechanics model in an averaged (continuum) sense, the model is being utilized where proven most accurate.

MCT is readily coupled with the finite element method. Consider again the examples described in section 4.1. Using a single finite element, MCT and the same loading, an average transverse tensile stress of 10 MPa can be
predicted as shown with the various RVE. Since MCT only involves calculation of a handful of equations at each integration point, the added computational burden is insignificant.

Table 2 shows the average matrix stress for a unidirectional composite subjected to unconstrained cooling from 20°C to -197°C and the run time as determined by two micromechanics models and the multicontinuum decomposition. Note that each technique gives the same results, but the multicontinuum decomposition runs more than 10 times faster than either micromechanics method.

<table>
<thead>
<tr>
<th></th>
<th>Hexagonal Packing</th>
<th>Random Packing #1</th>
<th>Multicontinuum Decomposition</th>
</tr>
</thead>
<tbody>
<tr>
<td>σ_{22m}</td>
<td>25.9 MPa</td>
<td>25.9 MPa</td>
<td>25.9 MPa</td>
</tr>
<tr>
<td>Model Run Time</td>
<td>2.1 Seconds</td>
<td>8.4 Seconds</td>
<td>0.2 Seconds</td>
</tr>
</tbody>
</table>

Note that the average constituent stress and strain information provided by MCT is readily available from micromechanics. It is the extreme efficiency and ease of marriage to the finite element method that makes MCT valuable for structural analysis of composites.

One of the key advantages of the multicontinuum approach is that it opens the door to applying physics-based theories that have been proven accurate for polymers directly to the matrix of the composite. Previous works by Nelson, Hansen, and Mayes [10], Nelson et al. [17], and Mayes and Hansen [9], have leveraged the constituent information gained from the decomposition to improve failure predictions for composites by applying a modified von Mises failure criteria to the matrix.

In these works, failure criteria are applied separately to the fibers and matrix in the composite to predict constituent-level failure. Upon failure, the material properties of the failed constituent are degraded, realistically simulating failure in the composite. This idea has been successfully transferred to other known matrix phenomenon. In the two subsections below, we discuss the application of physics-based theories to the matrix to successfully predict pressure-induced strength enhancement and fatigue life in composite materials.

5.1 Pressure Induced Strength Enhancement

As a feature of the First World Wide Failure Exercise [13], it has become well publicized that polymer matrix composites experience strengthening under combined transverse compressive and longitudinal shear loads. Although many researchers have developed heuristic approaches to capture this effect [18], using MCT one can apply physics-based relationships developed to capture the strengthening effect pressure creates on polymers directly to the polymer matrix of the composite.

In a submission to the Second World Wide Failure Exercise, Nelson, Hansen, and Mayes [10], introduce an approach based on polymer work of Hoppel et al. [19] and Hine et al. [20], to capture this composite strengthening effect by only altering the strength of the matrix. Specifically, the form of matrix shear strength hardening under hydrostatic compressive stress is assumed to be

\[ S_{12m} = S_{12m} - C \left( \sigma_{kkm} \right), \]

where \( \sigma_{kkm} \) represents the trace of the matrix stress tensor at the current loading and the constant \( C \) is a material parameter that describes the hardening.

Consider a case from WWFE-I, a glass/epoxy unidirectional composite lamina subjected to a combined transverse tensile/compressive and shear stress state. Note in Figure 5 the increase in shear strength in compression quadrant. Also shown are comparisons to MCT failure predictions with and without pressure induced strength enhancement.
Figure 5. Biaxial, $\sigma_y$:$\tau_{xy}$, failure envelope for a [0] lamina made from E-glass/LY556/HT907/DY063.

Notice the difference between the two MCT failure envelopes. The dotted line contains no strength enhancement, the solid line applies Equation 5 to MCT using $c=0.15$. Excellent correlations to the experimental results are achieved using a very simple application of known polymer behavior.

5.2 Fatigue Life Prediction

For more than three decades, researchers have sought to predict the fatigue behavior of fiber reinforced polymer composites [21], [22], [23], [24], [25]. A range of approaches have considered several different types of fatigue loading: constant amplitude, block loading, and spectral loading. Yet no approach has been satisfactorily applied to the entire range of problems that must be addressed. Moreover, because the majority of composite fatigue analyses use homogenized composite stresses/strains rather than constituent stresses/strains, direct application of relevant material physics is nearly impossible.

The method outlined here for composite fatigue life prediction overcomes this deficiency by applying material kinetics directly to constituent materials using MCT. Thus it is done in an extremely computationally efficient manner, is readily coupled with finite element analysis, and requires minimal material characterization for fatigue life predictions under complex load states.

Using constituent-averaged stresses obtained from the MCT approach, realistic physics can be applied at the constituent level. It has been widely observed that, with the exception of uniaxial tensile fatigue along the fiber direction, composite fatigue failure is driven by fatigue of the polymer matrix. Thus, physics-based theories are applied to the matrix constituent. One way to relate time-temperature-load behavior of a polymer is by using the kinetic theory of fracture [26], [27] to develop a rate equation for bond-breaking in the polymer that can be linked to a normalized damage variable $n$ by [28]

$$\frac{dn}{dt} = (n_0 - n) f_0 \exp \left( \frac{U - \gamma \sigma_{eff} (t)}{kT} \right), \quad n_0 = \frac{e}{e-1}. \quad (6)$$
where \( f_0 \) is the frequency of atomic oscillation, \( U \) is the activation energy, \( \gamma \) is the activation volume, \( k \) is the Boltzmann constant, and \( \sigma_{\text{eff}} \) is an effective matrix stress. The initial value of the damage variable is zero and its value is unity at the point of failure. Thus, Eq. (6) provides the framework to assess fatigue damage in a composite.

To demonstrate the utility of this approach, consider two sets of off-axis carbon-epoxy composite fatigue data. The first is taken from Awerbuch and Hahn [29], the second from Kawai et al. [30]. Only two parameters must be determined for each material in order to apply Eq. (6): \( U \) and \( \gamma \). (The value of \( f_0 \) is assumed to be \( 10^{13} \) s\(^{-1}\) following Zhurkov [26]. The effective stress was calculated in a manner similar to the methods used by Nelson et al. [10]. The values of \( U \) and \( \gamma \) were calculated from the 90° data and used to predict fatigue life for the other off-axis load cases.

The results of these predictions for the Kawai et al. [30] data and Awerbuch and Hahn data [29] are shown in Figures 6 and 7, respectively. In these figures, the points correspond to experimental data; the solid lines correspond to predicted fatigue life. Both figures demonstrate surprising accuracy, particularly given the scatter in the experimental data and the fact that only two parameters are used for the prediction.

![Off-axis fatigue of T800H/2500EP](image)

Figure 6. Fatigue life predictions compared with experimental data taken from Kawai et al. [30]. Solid curves represent predictions; points represent experimental data. Colors correspond to different off-axis uniaxial tension loads.
The results presented in Figures 6 and 7 are preliminary, but they demonstrate the viability of applying physics to composite materials. The less obvious advantage of this approach, but no less valuable, is that only two parameters are required for these predictions. Thus, relatively few tests are required to characterize composite fatigue.

5. Conclusions

The continuum hypothesis has been proven remarkably successful for structural analysis of homogenous materials. Micromechanics approaches have proven most accurate for determining the average response of materials. In order to conduct accurate analysis of composite structures, some detail about the composite's microstructure is required to accurately predict material response.

To provide accurate failure predictions of composite structures, a blending of continuum mechanics and micromechanics contributes to the concept of a multicontinuum. Considering the additional information provided by the multicontinuum approach gives the following noteworthy results:

1. The continuum hypothesis is maintained and extended to multiple materials, building on the success of previous generations of engineers.
2. Important interactions between constituents are captured, allowing increased analytical accuracy.
3. When used in conjunction with the finite element method, virtually zero computational burden is added.

Multicontinuum mechanics provides insight into constituent behavior in a composite without the potential inaccuracy due to microstructural idealization or the time penalty micromechanics approaches impose, making its use practical for the analysis of large structures. The information can be leveraged to predict material behavior in the constituent such as failure or fatigue. Moving forward, failure and fatigue predictions at this level offers significant improvements in accuracy and will be critical in understanding performance of large, complex composite structures.
Acknowledgements
The research was sponsored by the Air Force Research Laboratory (AFRL) under contract number FA 9453-07-C-0191 under the direction of Dr. Thomas Murphey.

References
APPENDIX A: Derivation of the Multicontinuum Concept

In order to develop the multicontinuum concept, consider a continuous fiber composite material where the matrix \((m)\) and fibers \((f)\) are allowed to retain their identity in the continuum. Using governing equation of continuum mechanics for each constituent one can write:

\[
\sigma_f = \frac{1}{V_f} \int_{D_f} \sigma(x) \, dV ,
\]

And

\[
\sigma_m = \frac{1}{V_m} \int_{D_m} \sigma(x) \, dV ,
\]

Where

\[
D = D_f \cup D_m .
\]

Combining Eqs. 1-3 leads to

\[
\sigma = \phi_f \sigma_f + \phi_m \sigma_m ,
\]

where \(\phi_f\) and \(\phi_m\) are the volume fractions of fiber and matrix, respectively. Likewise, for strains there follows

\[
\varepsilon = \phi_f \varepsilon_f + \phi_m \varepsilon_m .
\]

Changing from direct tensor to contracted matrix notation, linear elastic constitutive laws for the composite and the constituents are given by

\[
\{\sigma\} = [C] \left( \{\varepsilon\} - \{\varepsilon_o\} \right) ,
\]

\[
\{\sigma_f\} = [C_f] \left( \{\varepsilon_f\} - \{\varepsilon_{fo}\} \right) ,
\]

and

\[
\{\sigma_m\} = [C_m] \left( \{\varepsilon_m\} - \{\varepsilon_{mo}\} \right) .
\]

Combining Eqs. 4-8, constituent fiber and matrix strain fields, \(\{\varepsilon_f\}\) and \(\{\varepsilon_m\}\) respectively, are derived from the composite strain field \(\{\varepsilon\}\) using
\{\varepsilon_m\} = (\phi_m [I] + \phi_f [A])^{-1} \left(\{\varepsilon\} - \Delta T \{a\}\right), \quad (9)

and

\{\varepsilon_f\} = \frac{1}{\phi_f} \left(\{\varepsilon\} - \phi_m \{\varepsilon_m\}\right), \quad (10)

where

\[ A = -\frac{\phi_m}{\phi_f} ([C] - [C_f])^{-1} ([C] - [C_m]), \quad (11) \]

and

\{a\} = ([C] - [C_f])^{-1} \left(\{\alpha\} - \phi_f [C_f] \{\alpha_f\} - \phi_m [C_m] \{\alpha_m\}\right). \quad (12)