An Invariant-Based Flow Rule for Anisotropic Plasticity Applied to Composite Materials

In this paper we discuss some fundamental problems associated with incremental anisotropic plasticity theories when applied to unidirectional composite materials. In particular, we question the validity of an effective stress-strain relation for highly anisotropic materials of this nature. An effective stress-strain relation is conventionally used to determine a flow rule for plastic strain increments. It is our view that such a relation generally does not exist for many high-performance unidirectional composites. To alleviate the problem associated with defining an effective stress-strain curve we develop an anisotropic plasticity theory in which the flow rule does not require such a relation. The proposed theory relies on developing an accurate expression for a scalar hardening parameter $g(\sigma)$. The general form of $g(\sigma)$ is substantially reduced by invoking invariance requirements based on material symmetry. The general invariant-based theory developed herein is specialized to case of transverse isotropy and applied to the analysis of a nonlinear elastic-plastic unidirectional composite material. The invariant-based theory is shown to produce superior results over the traditional approach for a series of uniaxial and biaxial load cases predicted using finite element micromechanics.

Introduction

The mathematical theory of plasticity is based on the existence of a plastic potential or yield function which demarcates the material behavior from elastic to plastic. The yield function is, in general, dependent on the stress state, and perhaps on an internal state vector which characterizes the plastic state of the material. The stress state must lie on the yield surface in order for plastic deformation to occur. Furthermore, one can show, using the energy arguments, that the increment in plastic strain must be normal to the yield surface. Therefore, a general form of an invariant constitutive law for plastic strain increments assumes the form

$$\dot{\varepsilon}_{ij} = d\lambda \frac{\partial \Phi}{\partial \sigma_{ij}},$$

where $\Phi$ is the yield function. The scalar $d\lambda$ is normally determined by assuming there exists a unique effective stress-strain curve for the material such that the plastic work may be written as

$$dW^p = \sigma d\varepsilon^p.$$

One can show the value of $d\lambda$ is a function of the tangent modulus of the effective stress-strain relation. The form of the effective stress-strain curve is determined experimentally using a specific load path and is then assumed to be valid for any multiaxial loading. This assumption is the foundation of isotropic plasticity and has generally been observed to be true for a wide variety of metals.

The constitutive law given by equation (1) has been extended to anisotropic materials by several investigators. Perhaps the most well known of these is attributable to Hill (1950) who developed an orthotropic plasticity theory for cold-rolled steels. The theory has its roots in the isotropic formulation and uses an effective stress-strain relation to determine the specific value of $d\lambda$. This approach has been specialized to the case of transversely isotropic materials to investigate the behavior of unidirectional composite materials (Griffin et al., 1981). The theory developed by Hill (1950) is fundamentally sound and represents a major contribution to the theory of anisotropic plasticity. However, application of this theory to high-performance unidirectional composites must be questioned. In particular, we do not accept the concept of an effective stress-strain relation for highly anisotropic materials of this nature. The problem lies in the fact that the tangent modulus of the effective stress-strain relation is generally load-path dependent for this type of material. For instance, the behavior of many high-performance composites may be either linear elastic to failure or highly inelastic depending on the type of loading. The difference in behavior may be attributed to a variety of
deformation mechanisms occurring on the microscale which are load-path dependent. Hence, the question arises as to what load path should be used to determine the tangent modulus, and hence $d\lambda$, when the material is under multiaxial loads.

The lack of a unique effective stress-strain relation for anisotropic materials has been noted by several previous investigators. Kenaga et al. (1987) developed a two-dimensional orthotropic plasticity theory to predict the plane stress behavior of unidirectional boron/aluminum composites. An optimum effective stress-strain relation was determined for the material using a trial and error analysis of off-axis tension test data. Sun and Chen (1988) extended the work of Kenaga et al. (1987) by reducing the number of coefficients needed for the effective stress-strain relation from three to one. A drawback to these works is that the parameters used to trial and error fit the effective stress-strain relation also directly influence the yield function. This is theoretically overly restrictive in the sense that the yield function should not be influenced by the effective stress-strain relation. Furthermore, the procedure for extending the theory to fully three-dimensional stress states is unclear.

Gotoh (1977) assumed a yield function that is fourth order in stress in his investigation of cold-rolled steels. While not directly applicable, he makes an important observation noting that the tangent modulus of the "effective stress-strain" curve is in fact dependent on the type of loading, even though the curves should be intrinsically unique for a given material.

To alleviate problems associated with a flow rule for anisotropic plasticity we develop a plasticity theory in which the flow rule does not require an effective stress-strain relation. The constitutive law is cast in a form which sheds considerably more light on the specific nature of the flow rule. In particular, we reduce the problem to developing an accurate expression for a scalar-hardening parameter $g(\phi)$. One significant aspect of this theory is that the explicit value of the scalar $g(\phi)$ varies depending on the specific location of the stress state on the yield surface. This approach is in sharp contrast with the traditional approach of Hill in which $g(\phi)$ is a constant everywhere on the yield surface. This is an implicit result of assuming the existence of an effective stress-strain curve.

The general form of $g(\phi)$ can be substantially simplified by invoking invariance requirements on the material based on material symmetry. This is accomplished through the use of representation theorems for tensor functions. These theorems place valuable restrictions on the possible functional forms of tensor functions and are particularly useful when modeling anisotropic materials.

Two examples of the use of representation theorems for modeling anisotropic plasticity may be found in the work of Boehler (1987) and Spencer (1987). Boehler has developed an anisotropic-hardening theory for rolled sheet-steel whose macroscopic behavior is orthotropic. The constitutive law developed assumes the form

$$T = F(D, P, M)$$

where $T$, $D$, and $P$ are the stress, kinematic, and prestrain tensors, respectively. $M$ is a structural tensor which characterizes the initial orthotropy of the material. Invariant-based yield criterion and hardening rules are then formulated based on an irreducible representation of equation (3) using representation theorems for tensor functions.

Spencer (1987) has also developed a plasticity theory for anisotropic materials based on representation theorems. In this work, plastic strain increments are defined in a manner analogous to equation (1). However, the current state of hardening is assumed to depend on the history of the strain rather than the current stress as proposed in this work. Hence, scalar invariants of the strain tensor are defined and general theories of "proportional hardening" and kinematic hardening are developed.

Problems Associated with Anisotropic Incremental Theories

Here we discuss some problems associated with incremental anisotropic theories when applied to unidirectional composite materials. We assume the composite may be modeled as a transversely isotropic material. In doing so, we specialize the orthotropic theory proposed by Hill (1950) to discuss these problems. However, these same problems associated with the Hill theory are present in many of the modified theories in which the yield surface is altered in some manner.

Hill proposed that the simplest yield criterion for an anisotropic material is one that reduces to the von Mises yield criterion when the anisotropy is vanishingly small. In the spirit of Hill, a quadratic form of the six components of stress is chosen for a transversely isotropic yield function as

$$\Phi = \phi - \phi_0,$$  \hspace{1cm} (4)

where

$$\phi = F(\sigma_{22} - \sigma_{33})^2 + G(\sigma_{13} - \sigma_{12})^2 + G(\sigma_{11} - \sigma_{22})^2 + (G + 2F)(\sigma_{23}^2 + \sigma_{31}^2) + M(\sigma_{33}^2 + \sigma_{11}^2 + \sigma_{22}^2),$$  \hspace{1cm} (5)

where the $\chi_3$-axis represents the axis of rotational symmetry. The value $\phi_0$ represents the largest recorded value of $\phi$. For initial yield, $\phi$ is taken to be unity. This approach represents an isotropic hardening theory.

In the foregoing equations, $F$, $G$, and $M$ are parameters characterizing the degree of anisotropy. These parameters remain unchanged during deformation, consistent with an isotropic hardening theory.

As customary the yield function, $\Phi$, defines the following material behavior:

- $\Phi < 0$ - elastic behavior,
- $\Phi = 0$ - the stress state lies on the yield surface, and
- $\Phi > 0$ - inaccessible state.

Let $\sigma_{ij}'$ represent the initial yield stresses referenced to the orthogonal material coordinates. The parameters $F$, $G$, and $M$ in equation (5) can be solved as

$$2F = \frac{2}{(\sigma_{22}^2)} - \frac{1}{(\sigma_{11}^2)},$$

$$2G = \frac{1}{(\sigma_{22}^2)},$$

$$2M = \frac{1}{(\sigma_{33}^2)}. \hspace{1cm} (6)$$

The yield stresses $\sigma_{ij}'$ must be known from experimental data or estimated analytically, e.g., using finite element micromechanics.

In order for plastic deformation to occur, the state of stress must lie on the yield surface, i.e., $\Phi = 0$. Following Martin (1975), the increment in plastic strain may be written as

$$d\varepsilon_{ij}' = g(\phi) \left( \frac{\partial \Phi}{\partial \sigma_{ij}} d\sigma_{ij} \right) \frac{\partial \Phi}{\partial \sigma_{ij}'}.$$

when

$$\frac{\partial \Phi}{\partial \sigma_{ij}} d\sigma_{ij} > 0 \text{ and } \Phi(\sigma) = 0. \hspace{1cm} (7)$$

Comparing equations (1) and (7) we see that the scalar $d\lambda$ of equation (1) is given by

$$d\lambda = g(\sigma) \left( \frac{\partial \Phi}{\partial \sigma_{ij}} d\sigma_{ij} \right). \hspace{1cm} (8)$$

The significance of the latter term in the above becomes apparent by noting that $\partial \Phi/\partial \sigma_{ij}$ is a vector normal to the yield surface in stress hyperspace. Hence, $(\partial \Phi/\partial \sigma_{ij}) d\sigma_{ij}$ is a measure of the component of the incremental stress normal to the yield...
surface. The sign of this term determines the loading condition for the material as follows:

\[
\begin{align*}
(\partial \Phi / \partial \sigma_{ij}) d\sigma_{ij} >& 0 \quad \text{loading;} \\
(\partial \Phi / \partial \sigma_{ij}) d\sigma_{ij} =& 0 \quad \text{neutral loading;} \\
(\partial \Phi / \partial \sigma_{ij}) d\sigma_{ij} <& 0 \quad \text{unloading.}
\end{align*}
\]

From equation (8), we see that plastic strain increments can only occur during loading.

To complete the theory one must develop a functional form for the scalar-hardening coefficient \( g(\sigma) \). The simplest form of \( g(\sigma) \) is to assume it has the same value at any point on a given yield surface. This is the approach taken in isotropic plasticity. For instance, for a von Mises yield surface we can write

\[
\Phi = \frac{2}{3} (J_2 - \bar{J}_2),
\]

where \( J_2 \) represents the second invariant of the deviatoric stress and \( \bar{J}_2 \) represents the highest recorded value of \( J_2 \) beyond initial yield. A common form assumed for the scalar hardening parameter is given by

\[
g(\sigma) = g(J_2).
\]

The fact that \( g \) is only a function of the isotropic stress invariants is consistent with the representation theorems for a scalar function of a second-order tensor.

An experimental test for some particular load path is used to determine the specific functional form of \( g(J_2) \). The normal convention is to determine \( g(J_2) \) from a uniaxial tension test. For this case we arrive at a final form for \( g(J_2) \) given by

\[
g(J_2) = \frac{27}{16 J_2} \left( \frac{1}{E} - \frac{1}{E_0} \right),
\]

where \( E \) and \( E^T \) are the elastic modulus and tangent modulus from the uniaxial stress strain curve. The assumption that the form of \( g(J_2) \) as determined from a uniaxial test is valid for arbitrary load paths is central to the success of plasticity theory for isotropic materials. This in fact has been shown to be true for many materials; see, for example, Ivey's (1961) work on silicon-aluminum alloys. Furthermore, it is precisely this same assumption which leads to difficulties in modeling anisotropic material behavior.

Development of a functional form for the scalar-hardening coefficient in anisotropic plasticity follows the isotropic approach. For instance, the simplest approach is to assume \( g(\sigma) \) is a function of the current yield surface \( \Phi \). This is in fact the approach taken by Hill (1950). To demonstrate this we follow the work of Hill and write

\[
d\Phi = d\lambda \frac{\partial \Phi}{\partial \sigma_{ij}}
\]

The increment in plastic work is then given by

\[
dW^p = \sigma_{ij} d\Phi_{ij} = \sigma_{ij} d\lambda \frac{\partial \Phi}{\partial \sigma_{ij}}.
\]

Noting equations (4) and (5) and carrying out the required differentiation gives

\[
dW^p = 2d\lambda \sigma.
\]

To determine the constant \( d\lambda \) we introduce the concept of an effective stress for the material which satisfies

\[
dW^p = \sigma d\bar{\sigma}.
\]

Following the work of Hill (1950), the specific choice taken for \( \sigma \) is

\[
\bar{\sigma} = \sqrt{3} \left( \frac{F(x_{11} - x_{11})}{F + 2G} + \frac{2(G + 2F) x_{11}^2 + 2Mo_{1}^2 + 2Mo_{2}^2}{F + 2G} \right)^{1/2}.
\]

Comparing equations (5) and (17) it follows that

\[
\sigma^2 = \frac{3}{2} \frac{\phi}{F + 2G}.
\]

The plastic work, as defined by equation (16), may be rewritten as

\[
dW^p = \sigma \frac{d\bar{\sigma}}{d\sigma} d\sigma = \sigma d\sigma \frac{1}{H'},
\]

where

\[
H' = d\lambda \frac{d\bar{\sigma}}{d\sigma}.
\]

Differentiating equation (18) and substituting into equation (19) it follows that

\[
dW^p = \frac{3}{4(F + 2G)} \frac{1}{H'} \frac{d\sigma}{\partial \sigma_{ij}}
\]

Comparing equation (21) with equation (15), we obtain

\[
d\lambda = \frac{3}{8 \phi} \frac{1}{(F + 2G)} \frac{1}{H'}
\]

Finally, noting equation (9) we obtain a form for the scalar-hardening coefficient as a function of the yield surface given by

\[
g(\Phi) = \frac{3}{8 \phi} \frac{1}{(F + 2G)}
\]

The function \( (1/H') \) is determined by considering a specific load path. For instance, consider a uniaxial tension test in the \( x_1 \) direction. For this case the effective stress is

\[
\sigma = \left( \frac{3G}{F + 2G} \right) x_{11}.
\]

The effective strain is defined such that the increment in plastic work is energetically consistent with equation (14). Hence,

\[
d\bar{\sigma} = \left( \frac{F + 2G}{3G} \right) d\sigma_{11}.
\]

Noting equation (20), the function \( (1/H') \) is

\[
\frac{1}{H'} = \frac{1}{E_{11} - E_{11}} \frac{F + 2G}{3G},
\]

where \( E_{11} \) and \( E_{0} \) denote the modulus and the tangent modulus of the uniaxial stress-strain curve. Substituting equation (26) into equation (23), the scalar-hardening coefficient becomes

\[
g(\Phi) = \frac{1}{8 \phi} \frac{1}{E_{11} - E_{11}} \frac{1}{E_{11}}.
\]

Again, the fundamental assumption behind this approach is that the form for \( g(\Phi) \), determined from the uniaxial tension test, is valid for any multiaxial stress state. Herein lies a problem with the use of such a theory when applied to unidirectional composite materials in that the scalar-hardening coefficient is dependent on the specific location of the stress state on the yield surface. Hence, the assumption that the value of \( g(\Phi) \) is a constant for any given yield surface is overly restrictive. The specific behavior of the scalar-hardening coefficient for unidirectional composite materials will be examined using a finite element micromechanics analysis.
Fig. 1 Finite element mesh used for the fiber-matrix micromechanics model

Fig. 2 Uniaxial tension stress-strain plot of a ductile matrix material

Fig. 3 Micromechanics generated stress-strain curves for longitudinal tension, transverse tension, and longitudinal shear loadings

Fig. 4 Scalar-hardening coefficient plotted as a function of the yield surface for transverse tension and longitudinal shear load paths

represents a viable alternative to extensive experimental characterization of materials, particularly for multiaxial behavior. For instance, parametric studies involving different fiber volumes and constituent properties are readily carried out, thereby allowing one to characterize a wide variety of materials under various loading conditions.

The finite element micromechanics analysis used in this investigation is a generalized plane-strain analysis of a quarter fiber and surrounding matrix, representative of a continuous fiber unidirectional composite material, as shown in Fig. 1. The fiber direction is taken to be \( x_1 \). A square packing array is assumed for the fibers.

For example purposes, we choose to model the fiber as a stiff transversely isotropic material which is linear elastic to failure. The matrix material is softer than the fiber and is assumed to behave elastically plastically. The uniaxial stress-strain curve for the matrix constituent is taken to be bilinear as shown in Fig. 2. The specific elastic coefficients for the fiber are:

\[
E_{11} = 417.0 \text{ GPa}, \quad E_{22} = E_{33} = 208.5 \text{ GPa}, \quad G_{12} = G_{13} = G_{23} = 83.4 \text{ GPa}, \quad \nu_{12} = \nu_{13} = 0.2, \quad \nu_{23} = 0.25.
\]

The constituent behavior described above is not intended to model any specific material. Rather, the intent is to demonstrate the fundamental problems associated with modeling highly anisotropic materials of this nature. However, it should be noted that the ratios of the elastic moduli are typical of those found in many high performance composites such as boron/aluminum and silicon-carbide/titanium.

Figure 3 represents the behavior of the composite material as predicted by a micromechanics finite element analysis. The figure shows longitudinal tension (\( \sigma_{11} \)), transverse tension (\( \sigma_{22} \)), and longitudinal shear (\( \sigma_{12} \)) loadings. Note in Fig. 3 the longitudinal tension curve is not shown in its entirety since it is elastic to failure. The data clearly indicate the value for the scalar-hardening coefficient is strongly dependent on which loading is used to define \( g(\phi) \). For instance, the longitudinal tension stress-strain curve is essentially linear elastic to failure resulting in no plastic strain. Hence, a value of \( g(\phi) = 0 \) is indicated. In contrast, the behavior of the same composite material subjected to shear loads is highly nonlinear, indicating a value \( g(\phi) > 0 \). Furthermore, the value of \( g(\phi) \) is different for various loadings which exhibit plastic deformation. For instance, consider the longitudinal shear and transverse tension curves. Noting equation (23) the value of \( g(\phi) \) may be determined for each load case as:

\[
\sigma_{12}: \quad g(\phi) = \frac{1}{G_{12}} = 1/G_{12} / (8 \phi M) \tag{28}
\]

\[
\sigma_{22}: \quad g(\phi) = \frac{1}{E_{22}} = 1/E_{22} / (4 \phi (F + G)) \tag{29}
\]

where \( G_{12} \) denotes the shear modulus and \( E_{22} \) represents the elastic modulus in the \( x_2 \)-direction.
Figure 4 shows a plot of $g(\phi)$ for the transverse tension and longitudinal shear loadings as determined by equations (28) and (29). The figure clearly indicates a difference in the behavior of the scalar-hardening coefficient. This is also evidenced in Fig. 3 where it is seen that the longitudinal shear behavior is much more nonlinear than the transverse tension data. The question then arises as to what relation to use for $g(\phi)$ under multiaxial load cases.

**An Invariant-Based Flow Rule**

In this section we develop an advanced anisotropic plasticity theory in which the scalar-hardening coefficient is allowed to be load-path dependent. For comparison purposes, we assume a yield function identical to the one specified in equations (4) and (5). Furthermore, the plastic strain increments are assumed to be governed by equations (7) and (8), repeated for convenience as

$$\Delta e_{ij} = g(\phi) \left( \frac{\partial \phi}{\partial \sigma_{nn}} \delta_{nn} \right) \frac{\partial \phi}{\partial \sigma_{ij}},$$  \hspace{1cm} (30)

when

$$\frac{\partial \phi}{\partial \sigma_{nn}} \delta_{nn} > 0 \quad \text{and} \quad \Phi(\sigma) = 0.$$  \hspace{1cm} (31)

At this point we turn our attention to the scalar-hardening coefficient $g(\phi)$. We note that $g$ is a scalar function of a second-order tensor. The functional form of $g(\phi)$ may be restricted by considering invariance properties of the material. For example, in modeling a unidirectional composite material, the usual approach is to assume the material is transversely isotropic. Under this assumption the functional form of $g(\phi)$ must remain unchanged for arbitrary rotations about the axis of symmetry. Denoting this axis as $x_1$, the five transversely isotropic stress invariants for such coordinate orientations are given by Spencer (1971) as:

$$a_1 = a_{11}, \quad a_2 = a_{22} + a_{33}, \quad a_3 = a^2_{12} + a^2_{13} + 2a^2_{23},$$

where

$$a_4 = a^2_{12} + a^2_{13}, \quad a_5 = a_{22}a_{33} - a^2_{23} + 2a_{12}a_{13},$$  \hspace{1cm} (32)

A mathematically correct representation of $g(\phi)$ must be of the form

$$g(\phi) = g(a_1, a_2, a_3, a_4, a_5).$$  \hspace{1cm} (33)

In order to further reduce equation (33), we must make some additional assumptions about the material behavior. First, we require the value of $g(\phi)$ to remain unchanged when the signs of $a_{ij}$ are reversed. This implies that we must deal with quadratic forms for $a_1, a_2,$ and $a_3$ as these invariants are odd functions of stress. We reject the fifth invariant for simplicity in that $a^2_5$ is sixth order in stress. A further simplification is achieved by rejecting the second invariant $a_2$. In doing so, we are restricting the nonlinear behavior of the model in the sense that the behavior of $g(\phi)$ will be identical for both transverse tension and transverse shear load paths. The micromechanics analysis used in this paper was unable to verify this as a square packing geometry is not truly transversely isotropic. However, it should be noted that the invariant formulation is capable of modeling differences in these load paths simply by incorporating the second invariant in the function form of $g$. Therefore, a reduced form for $g(\phi)$ is

$$g(\phi) = g(a_1, a_2, a_3).$$  \hspace{1cm} (34)

Finally, one may be tempted to eliminate the first invariant $a_1$ as the material is linear elastic to failure under uniaxial tension in the fiber direction. However, longitudinal tensile stresses do influence the nonlinear behavior of the material under multiaxial loads. This can only be accounted for by retaining $a_1$.

To complete the theory a specific functional form for $g(a_1, a_2, a_3)$ must be developed using experimental and/or micromechanics data. The specific form will be material dependent.

**Fig. 5** Scalar-hardening coefficient plotted as a function of the stress invariants $a_3$ and $a_4$ for transverse tension and longitudinal shear load paths, respectively.

To begin, we consider specific load paths in which the invariants $a_1$ and $a_2$ may be isolated. To study the effects of $a_3$ alone we consider a longitudinal shear test in which $a_{12} = 0$ and all other $a_{ij} = 0$. Rewriting equation (29) as a function of the fourth invariant gives

$$a_{12}: \quad g(\phi) = (1/G_{12} - 1/G_{12})/(16 a_4 M^2).$$  \hspace{1cm} (35)

The third invariant may be isolated by considering a transverse tension test in the $x_2$-direction. For this we can rewrite equation (29) to obtain

$$a_{22}: \quad g(\phi) = (1/E_{22} - 1/E_{22})/(4 a_4 (F + G)^2).$$  \hspace{1cm} (36)

A linear regression analysis of the data yielded the following forms for $g_3$ and $g_4$:

$$g_3 = 1.2 \times 10^{-6} a_1 - 1.2 \times 10^{-4} \text{ MPa}$$

$$g_4 = 3.0 \times 10^{-5} a_4 - 0.5 \text{ MPa}.$$  \hspace{1cm} (37, 38)

At this point one must assume a functional relationship for the scalar-hardening coefficient under multiaxial loadings. The form chosen must reduce to the specific forms defined by equations (37) and (38) for the corresponding loading conditions. Furthermore, the influence of longitudinal stress on the nonlinear behavior must be incorporated. A negligible coupling effect between $a_{11}$ and $a_{22}$ was observed in biaxial micromechanics analyses. However, there was significant coupling between $a_{12}$ and $a_{13}$ in biaxial runs. Therefore, the final form for the scalar-hardening coefficient was taken as

$$g(a_1, a_2, a_3) = \frac{a_3}{a_3} g_3(a_3) + \frac{a_4}{a_4} (g_4(a_4) + g_4(a_1))$$  \hspace{1cm} (39)

where $a_3^*$ and $a_4^*$ are values of the stress invariants at the current yield surface as determined from a uniaxial transverse tension and a longitudinal shear load case, respectively. The influence of the first invariant was determined from a single biaxial micromechanics test in which $a_{11} = 10a_{12}$. The specific functional form taken for $g_1(a_1)$ is

$$g_1(a_1) = 6.5 \times 10^{-5} a_1^2 - 0.5 \times 10^{-4} a_1.$$  \hspace{1cm} (40)

It should be noted that the coefficients of equation (40) are several orders of magnitude smaller than those of equation (38). This is consistent with the notion that longitudinal tension stresses do not have a major influence on the yielding of the material. However, the work performed here indicates that these stresses cannot be totally neglected.

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The form of the scalar-hardening coefficients shown in equation (39) is now assumed to be valid for general three-dimensional loadings. While the approach used to determine a form for the scalar-hardening coefficient is not general, it does indicate the salient feature of the invariant-based theory in that $g(\phi)$ is allowed to vary on the yield surface. This may be contrasted with the classical theory in which $g(\phi)$ is a constant everywhere on the yield surface.

Results

In this section we compare the invariant-based theory and the classical theory based on the work of Hill (1950) for a transversely isotropic material using a series of uniaxial and biaxial load cases predicted by finite element micromechanics. It should be noted that all shear stress-strain data is presented graphically using the engineering definition of strain as is customary. However, the equations developed in the previous section are based on the tensorial definition of shear strain to take full advantage of indicial notation.

To begin, one must choose a specific load path to determine the scalar-hardening coefficient $g(\phi)$ to be used in the Hill formulation. For example purposes, we shall determine $g(\phi)$ using the longitudinal shear data shown in Fig. 3. This is a reasonable approach in that $\sigma_{12}$ is the stress for which the most pronounced nonlinearity is observed. To choose the longitudinal tension curve, as is traditional for isotropic plasticity theory, would indicate $g(\phi)$ as zero. This would suppress all plastic strains for any applied load path. This fact alone should cause one to question the validity of an effective stress-strain curve for anisotropic materials of this type. Finally, because the longitudinal tension behavior of the material is linear elastic to failure, the yield stress is defined as the ultimate stress for this loading.

One could argue that an optimum effective stress-strain curve could be chosen based on averaging all available data in order to minimize the error. This was the approach taken by Kenaga et al. (1987) for a biaxial test program. However, for fully multiaxial stress states such a procedure is unclear. Furthermore, as will be demonstrated, one can not totally eliminate the errors in this manner as the error is not consistent.

Figures 6 and 7 compare the longitudinal shear ($\sigma_{12}$) and uniaxial transverse tension ($\sigma_{22}$) numerical micromechanics predictions with the invariant-based theory and the classical formulation put forth by Hill. As expected, the two theories fall extremely close to the micromechanics prediction for longitudinal shear loading as shown in Fig. 6. However, the results are markedly different for transverse tension loading depicted in Fig. 7. For this case, the invariant-based theory tracks the micromechanics predictions while the Hill formulation predicts a much softer response. The discrepancy in the value predicted by the Hill formulation may be attributed to $g(\phi)$ being a constant for the entire yield surface.

A series of biaxial as opposed to uniaxial loadings were also modeled using finite element micromechanics to further compare the two plasticity theories. In each case the loading was assumed to be proportional and monotonically increasing. Figure 8 presents a $\sigma_{11} - \sigma_{12}$ stress-strain curve for a combined longitudinal tension ($\sigma_{11}$) and longitudinal shear ($\sigma_{12}$) loading where $\sigma_{11} = 10 \sigma_{12}$. The numerically predicted behavior of the material is near linear elastic. Both the invariant-based and Hill formulations agree quite well with the numerical results.

The theory developed by Hill works well in predicting this behavior because the anisotropic yield parameter $G$ is very small, tending to suppress incremental plastic strain values for this loading mode. Figure 9 shows the $\sigma_{12} - \gamma_{12}$ response for the same biaxial loading ($\sigma_{11} = 10 \sigma_{12}$) case. Here, the invariant formulation performs slightly better than the Hill formulation although both theories are close to the micromechanics response. However, it is worth noting that the Hill formulation is conservative for this case, which is in sharp contrast to other biaxial cases. This is especially interesting in that $g(\phi)$ was determined from shear data showing the greatest degree of nonlinearity.
Fig. 9 Comparison of $c_{11}$ stress-strain predictions for a combined longitudinal tension and longitudinal shear loading where $\sigma_{11} = 10 \sigma_{12}$

Fig. 10 Comparison of $c_{11}$ stress-strain predictions for a combined longitudinal tension and transverse tension loading where $\sigma_{11} = 5\sigma_{22}$

Fig. 11 Comparison of $c_{22}$ stress-strain predictions for a combined longitudinal tension and transverse tension loading where $\sigma_{11} = 5\sigma_{22}$

Fig. 12 Comparison of $c_{12}$ stress-strain predictions for a combined transverse tension and longitudinal shear loading where $\sigma_{22} = 2\sigma_{12}$

Fig. 13 Comparison of $c_{22}$ stress-strain predictions for a combined transverse tension and longitudinal shear loading where $\sigma_{22} = 2\sigma_{12}$

Fig. 14 Comparison of $c_{22}$ stress-strain predictions for a biaxial loading given by $\sigma_{22} = \sigma_{33}$

Figures 10 and 11 compare the plasticity theories for a biaxial loading combining longitudinal tension ($\sigma_{11}$) and transverse tension ($\sigma_{22}$) where $\sigma_{11} = 5\sigma_{22}$. The longitudinal tension data shown in Fig. 10 again reflect a linear response for all three curves. However, the transverse tension stress-strain response, Fig. 11, shows the invariant formulation is superior to the classical theory in following the numerical micromechanics results.

Similar results were found for biaxial analyses involving transverse tension ($\sigma_{22}$) and longitudinal shear ($\sigma_{12}$) as shown in Figs. 12 and 13. The load path was taken to be $\sigma_{22} = 2\sigma_{12}$. For the longitudinal shear results plotted in Fig. 12, the in-
variational formulation tracks the micromechanics analysis while the Hill formulation is significantly softer. The transverse tension results plotted in Fig. 13 are interesting in that both theories show a softer response than that predicted by micromechanics. However, the invariant based theory represents a substantial improvement when compared to the Hill formulation.

Finally, Fig. 14 shows a comparison of the two theories with micromechanics for a biaxial plot where $\sigma_{22} = \sigma_{13}$. In this case both theories perform extremely well.

In summary, the correlation of the invariant-based formulation with the predicted micromechanics results is excellent. In contrast, the Hill formulation worked well for some load paths while breaking down for others. The improved results of the invariant based theory may be attributed to allowing the value of the scalar-hardening coefficient to vary over the yield surface. The results are particularly pleasing in that a relatively simple form of the scalar-hardening function was assumed.

Discussion

In this paper we have developed a generalized anisotropic plasticity theory using an invariant-based flow rule. In particular, we allow the value of the scalar-hardening coefficient to vary depending on the specific location of the stress state on the yield surface. The functional form of the scalar-hardening function is developed in terms of the stress invariants and without the assumption of an effective stress-strain relation. The specific form of $g(\sigma)$ may be significantly reduced by invoking invariance requirements on $g$ based on material symmetry. This formulation permits more accurate modeling of uniaxial and multiaxial load cases without imposing the overly restrictive requirement of an effective stress-strain relation.

Development of an invariant-based flow rule has been slowed in the past by the specific requirement for an effective stress-strain relation. As has been demonstrated, such a relation generally does not exist for high performance, unidirectional composite materials. The assumed existence of an effective stress-strain relation implies the scalar-hardening coefficient is constant everywhere on the yield surface. Mathematically, this is an overly restrictive assumption in that $g(\sigma)$ is in general a function of the stress state. However, this fact is not immediately observable unless one casts the constitutive law in the form shown by equation (7).

Finally, we note that the invariant based theory developed here is based on an isotropic hardening model. For structures subjected to cyclic loading or significantly varying load paths it may be necessary to develop a kinematic hardening model. We refer to the excellent experimental work on metal matrix composites conducted by Dvorak et al. (1988) as an example.

Acknowledgments

This research was sponsored by the Honeywell Corporation, Defense Systems Division (A.C. Hansen) and Martin Marietta Corporation (D. M. Blackketter and D. E. Wairath).

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