A Plasticity Theory for Transversely Isotropic Materials

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A plasticity theory for transversely isotropic materials, based upon stress invariants, is implemented in a general finite element system and a modification is proposed that improves performance for certain multiaxial stress states. Performance of the modified invariant theory relative to Hill's theory is measured.

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A Plasticity Theory for Transversely Isotropic Materials
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Abstract
A fundamental difficulty with the use of Hill's plasticity theory for anisotropic materials is the need to select a unique effective stress–effective strain relation when none truly exists. Recently, an alternative theory based upon stress invariants which does not require definition of an effective stress–effective strain relation has been proposed. In this study, the invariant–based theory is implemented in a general finite element system and a modification is proposed that improves performance for certain multiaxial stress states. Performance of the modified invariant theory relative to Hill's theory is measured. Comparisons are based upon finite element micromechanics analysis and experimental test results. The invariant–based theory is shown to be superior to Hill's theory for a variety of uniaxial and multiaxial loading conditions.

Introduction
At present, there is no generally accepted constitutive model that can be used with confidence to simulate the nonlinear response of anisotropic materials under a variety of loading conditions. The most frequently used approach of representing the constitutive relations for anisotropic materials is Hill's incremental plasticity theory [Hill, 1950]. It works well for mildly anisotropic materials, such as cold rolled steels. However, when it is applied to high performance unidirectional composites, the common assumption that the effective stress–effective strain relation is governed by a single, uniaxial stress–strain relation has been shown to be overly restrictive [Hansen et al., 1991]. To avoid using a particular effective

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stress–effective strain relation, Hansen et al. developed a theory based upon stress invariants which has been shown to be superior to Hill’s theory. But for some particular cases, this new approach may also exhibit serious errors.

Several attempts have been made to define a single effective stress–effective strain relation for anisotropic materials that accounts for the different material properties in the different material directions. For determining this relation, Biffle [1973] used an assumption of equal plastic work in all directions. Piško et al. [1975] used the method of weighted averages. Kenaga et al. [1987] used a trial and error analysis of off–axis tension test data. Sun & Chen [1989] advanced the theory developed by Kenaga et al. by incorporating the fact that for most unidirectional fibrous composites, the stress–strain relation in the fiber direction is basically linearly elastic. The plastic behavior of composites was then characterized by a single parameter. However, three-dimensional stress states or complex material geometries were not addressed.

In the present paper, the invariant–based theory developed by Hansen et al. [1991] is reviewed and a modification is offered which tends to eliminate a defect in predicting response under certain multiaxial stress states. The modified invariant theory is then compared to Hill’s theory. The primary basis of comparison is finite element micromechanics analysis of an arbitrarily selected boron–aluminum composite. Experimental results from an off–axis tension test of a boron–epoxy laminate are also used to evaluate the two theories. Time–dependent loading, thermal loading, and geometrical nonlinearities are not treated.

**Plasticity Theories for Anisotropic Materials**

In what follows we will discuss some basic concepts and problems associated with incremental anisotropic plasticity theories when applied to transversely isotropic materials. The theories to be examined are Hill’s formulation and Hansen’s invariant–based formulation.
Hill's Formulation

The yield criterion proposed by Hill [1950] was for general orthotropic materials. For a transversely isotropic material (rotational symmetry about the 1–axis), a quadratic form of the yield function is derived as

\[
\Phi = \phi - \overline{\phi}
\]  

(1)

where

\[
\phi(\sigma) = F(\sigma_{22} - \sigma_{33})^2 + G(\sigma_{33} - \sigma_{11})^2 + G(\sigma_{11} - \sigma_{22})^2 + (G + 2F)(\sigma_{23}^2 + \sigma_{32}^2) + M(\sigma_{13}^2 + \sigma_{31}^2 + \sigma_{12}^2 + \sigma_{21}^2)
\]  

(2)

In Eq. (2), \(\sigma_{11}\), \(\sigma_{22}\), and \(\sigma_{33}\) are normal stresses in the direction of principal anisotropic axes 1, 2 and 3, respectively; \(\sigma_{12}\), \(\sigma_{13}\), and \(\sigma_{23}\) are the shear stresses in the 1–2, 1–3, and 2–3 planes, respectively; \(\overline{\phi}\) represents the largest recorded value of \(\phi\). For initial yield, \(\overline{\phi}\) is unity. Isotropic hardening is assumed.

In Eq. (2), the quantities \(F\), \(G\), and \(M\) are characteristic parameters of anisotropy. They are assumed as constant and can be determined directly from a series of uniaxial load test as

\[
2F = \frac{2}{Y^2} - \frac{1}{X^2}, \quad 2G = \frac{1}{X^2}, \quad 2M = \frac{1}{S_{12}^2}
\]  

(3)

where \(X\), \(Y\) denote the uniaxial yield strengths corresponding to the 1 and 2 axes, and \(S_{12}\) is the yield strength for shear loading in the 1–2 coordinate plane.
If the analysis is restricted to an associated flow rule, the relation between the plastic strain increment tensor $\mathbf{d}\varepsilon^p_{ij}$ and the stress tensor $\sigma_{ij}$ is derived from Drucker’s postulate, a consequence of which is that the plastic strain increment vector is normal to the loading surface

$$
\mathbf{d}\varepsilon^p_{ij} = d\lambda \frac{\partial \phi}{\partial \sigma_{ij}}
$$

(4)

Following Martin [1975], we write Eq. (4) as

$$
\mathbf{d}\varepsilon^p_{ij} = g \left[ \frac{\partial \phi}{\partial \sigma_{rs}} d\sigma_{rs} \right] \frac{\partial \phi}{\partial \sigma_{ij}}
$$

(5)

where the term $\frac{\partial \phi}{\partial \sigma_{rs}} d\sigma_{rs}$ serves as a loading/unloading indicator; $\frac{\partial \phi}{\partial \sigma_{rs}} d\sigma_{rs} > 0$, $\frac{\partial \phi}{\partial \sigma_{rs}} d\sigma_{rs} = 0$, and $\frac{\partial \phi}{\partial \sigma_{rs}} d\sigma_{rs} < 0$ represent loading, neutral loading and unloading, respectively. In general, the scalar hardening coefficient $g$ may depend on stress, strain, and history of loading. The simplest form of $g$ is to assume that it depends only on the value of yield function $\phi$. Hence, we write the scalar hardening coefficient as $g(\phi)$. The introduction of effective increment plastic strain $\mathbf{d}\bar{\varepsilon}_p$ and effective stress $\bar{\sigma}_e$ simplifies evaluation of $g(\phi)$.

For transverse isotropy, $\bar{\sigma}_e$ and $\mathbf{d}\bar{\varepsilon}_p$ are defined as

$$
\bar{\sigma}_e = \sqrt{\frac{3}{2(F + 2G)}} \left[ F(\sigma_{22} - \sigma_{33})^2 + G(\sigma_{33} - \sigma_{11})^2 + G(\sigma_{11} - \sigma_{22})^2 + 2(G + 2F)\sigma_{23}^2 + 2M(\sigma_{13}^2 + \sigma_{12}^2) \right]^{\frac{1}{2}}
$$

(6)
\[ d\xi_p = \sqrt{\frac{2}{3} (F + 2G)} \left[ \frac{(d\xi_{23})^2}{2(G + 2F)} + \frac{(d\xi_{31})^2}{2M} + \frac{(d\xi_{12})^2}{2M} \right. \\
\left. + \frac{FG(d\xi_{22} - d\xi_{33})^2 + (Gd\xi_{33} - Fd\xi_{11})^2 + (Fd\xi_{11} - Gd\xi_{22})^2}{G(2F + G)^2} \right]^{\frac{1}{2}} \]  

(7)

where engineering shear strains are used. By definition, the increment in plastic work is

\[ dw^p = \sigma_{ij} d\xi_{ij} \]  

(8)

Substitution of the plastic strain increment from Eq. (4) into Eq. (8) and combination with Eq. (2) gives

\[ dw^p = \frac{2}{3} \sigma_c d\lambda = 2\phi d\lambda \]  

(9)

If we assume the existence of a single effective stress–effective strain relation, \( dw^p \) can also be written as

\[ dw^p = \sigma_p d\xi_p \]  

(10)

The slope of effective stress versus effective plastic strain curve is defined as

\[ H' = \frac{d\sigma_c}{d\xi_p} = \frac{E_T E}{E - E_T} \]  

(11)

where \( E \) is the effective modulus of elasticity and \( E_T \) is the effective tangent modulus.

Equating Eq. (9) to Eq. (10) and using Eqs. (2), (6), and (11), we obtain \( d\lambda \) as

\[ d\lambda = \frac{3}{8\phi} \frac{1}{(F + 2G)} \frac{1}{H'} \frac{\partial \phi}{\partial \sigma_{ij}} d\sigma_{ij} \]  

(12)

Recall that \( d\lambda \) is related to \( g(\phi) \) by

\[ d\lambda = g(\phi) \frac{\partial \phi}{\partial \sigma_{ij}} d\sigma_{ij} \]  

(13)
Comparing Eqs. (12) and (13), we obtain \( g(\phi) \) as

\[
\frac{3}{8\phi (F + 2G)} \cdot \frac{1}{H'} = \frac{3}{8\phi(F + 2G)} \cdot \frac{E - E_T}{EE_T}
\]

(14)

The computation of \( g(\phi) \) is complicated by the presence of the term \( H' \). In Hill's formulation, \( H' \) is determined from an effective stress–effective strain diagram which is usually identified with one of the stress–strain curves along the principal anisotropic directions. The model formulation and the resulting predictions are also assumed to be independent of the choice of the prescribed stress–strain curve. However, numerical and experimental evidence indicate that the scalar hardening coefficient is generally load–path dependent for high–performance unidirectional composites. The specific behavior of the scalar hardening coefficient will be examined using a finite element micromechanics analysis.

**Material Property Characterization**

Finite element micromechanics analysis can be used to determine the properties of composite materials from the properties of constituent materials. In this study, the micromechanics model shown in Figure 1 is used. This geometric model represents a cross section of two quarter–fibers with surrounding matrix material. Generalized plane strain conditions are imposed to model a unidirectional composite with continuous fibers. The fiber direction is taken to be the 1 direction. This mesh assumes a hexagonal packing array for the fibers with a fiber volume fraction of 0.60. This approach allows one to characterize a wide variety of constituent materials under various loading conditions.

For example purposes, we consider a composite composed of transversely isotropic and linearly elastic fibers, surrounded by an elastic–linear hardening isotropic matrix material. The properties of the matrix material are

\[ E = 69 \text{ GPa}, \quad E_T = 0.35 \text{ GPa}, \quad v = 0.33, \quad X = 97 \text{ MPa} \]

The properties of the fiber are
\[ E_{11} = 417.0 \text{ GPa}, \quad E_{22} = E_{33} = 208.5 \text{ GPa}, \]
\[ G_{12} = G_{13} = G_{23} = 83.4 \text{ GPa}, \]
\[ \nu_{12} = \nu_{13} = 0.2, \quad \nu_{23} = 0.25 \]

These material properties are typical values for a boron–aluminum composite.

For isotropic materials, the effective stress–effective strain relation, determined from a single uniaxial test, is valid for any load case. Figure 2 contains the results of finite element micromechanics analysis of the example composite subjected to three uniaxial loading conditions. For an anisotropic material such as that considered here, it is clear that the behavior of the composite is strongly dependent on which loading path is used to define the effective stress–effective strain relationship. For instance, if the longitudinal tension stress–strain curve is selected for the effective stress–effective strain relation, we find that \( E_T \approx E \). Hence, the material remains essentially elastic for all load cases. In contrast, if the transverse tension or longitudinal shear stress–strain curve is chosen, \( E_T < < E \) and substantial plastic strains can be developed. Thus, selection of a single effective stress–effective strain curve to represent all plastic flow conditions is not valid for highly anisotropic composites.

**Invariant–Based Formulation**

To avoid the problems associated with selecting an effective stress–effective strain relation, Hansen *et al.* [1991] developed an invariant–based flow rule in which the scalar hardening coefficient is allowed to vary with the stress state for a given yield surface. If the material possess rotational symmetry about the 1–axis, meaning that the material is transversely isotropic, there are five stress invariants [Spencer, 1971]

\[
\begin{align*}
a_1 &= \sigma_{11}, & a_2 &= \sigma_{22} + \sigma_{33}, & a_3 &= \sigma_{22}^2 + \sigma_{33}^2 + \sigma_{23}^2, \\
a_4 &= \sigma_{12}^2 + \sigma_{13}^2, & a_5 &= \sigma_{22}\sigma_{12}^2 + \sigma_{33}\sigma_{13}^2 + 2\sigma_{12}\sigma_{13}\sigma_{23}
\end{align*}
\]

(15)
Based solely upon the invariant properties of the material, a general form of $g(\bar{\sigma})$, was proposed as

$$g(\bar{\sigma}) = g(a_1, a_2, a_3, a_4, a_5)$$  \hspace{1cm} (16)

This scalar hardening function is assumed dependent upon the entire stress state. Hence, hardening is assumed to be controlled by the position on the yield surface, not just the characteristic dimension (e.g. radius) of the yield surface. The number of invariants in Eq. (16) can be reduced by imposing restrictions on the behavior of the material. First, yield behavior of the material under tension or compression is assumed to be the same, thus, the odd term $a_5$ is eliminated. Second, in their original work, Hansen et al. were unable to test their micromechanics model in transverse shear, so they were unable to distinguish between $a_2$ and $a_3$. Consequently, $a_2$ was neglected. The resulting hardening function is

$$g(\bar{\sigma}) = g(a_1, a_3, a_4)$$  \hspace{1cm} (17)

Empirical data for $g(a_1, a_3, a_4)$ can be found from experimental tests and/or micromechanics analysis of a particular set of constituent materials. In addition, by examining various uniaxial stress conditions, functional forms for $g(\bar{\sigma})$ can be derived. For example, in longitudinal shear, the invariants $a_1$ and $a_3$ are zero. Hence, only invariant $a_4$ has effect on the scalar hardening coefficient. For this stress state, Eq. (17) reduces to

$$g(\bar{\sigma}) = g_4(a_4)$$  \hspace{1cm} (18)
Equations (6), (7), and (11) are reduced similarly to obtain

\[ d\bar{\sigma}_e = \left[ \frac{3M}{F + 2G} \right]^{\frac{1}{2}} d\sigma_{12} \]

\[ d\bar{\epsilon}_p = \left[ \frac{2(F + 2G)}{3} \cdot \frac{1}{2M} \right]^{\frac{1}{2}} dy_{12} \]

\[ \frac{1}{H'} = \frac{d\bar{\epsilon}_p}{d\bar{\sigma}_e} = \frac{F + 2G}{3M} \cdot \frac{1}{H'_{12}} \]

From Eqs. (2) and (14), \( g_4(a_4) \) is

\[ g_4(a_4) = \frac{1}{16a_4M^2} \cdot \frac{1}{H'_{12}} \tag{19} \]

where

\[ H'_{12} = \frac{d\sigma_{12}}{dy_{12}} \]

For a transverse tension load

\[ g(\bar{\sigma}) = g_3(a_3) \]

Using procedures similar to those for deriving \( g_4(a_4) \), we obtain the form of \( g_3(a_3) \) as

\[ g_3(a_3) = \frac{1}{4a_3(F + G)^2} \cdot \frac{1}{H'_{22}} \tag{20} \]

where

\[ H'_{22} = \frac{d\sigma_{22}}{d\bar{\epsilon}_{22}} \]

As originally proposed, the scalar hardening function was not evaluated for loading in longitudinal tension alone, since, for the material considered, the composite remained essentially elastic in over the stress range of interest. However, it was observed that
longitudinal tension did affect yielding in longitudinal shear. So the first invariant \(a_1\) was included with \(a_4\) to develop biaxial coupling. To test the invariant–based theory, Hansen et al. [1991] derived specific forms for the scalar hardening function from results of micromechanics analysis of an arbitrarily chosen composite material with a square packing geometry. Numerical studies indicated that \(g_3\) varied almost linearly with \(a_3\), \(g_4\) was almost linear in \(a_4\) and \(g_1\) approached a quadratic function of \(a_1\). Hence, a regression fit of the data for the numerical tests was used to describe the uniaxial scalar hardening functions \(g_1(a_1)\), \(g_2(a_3)\), and \(g_4(a_4)\) instead of the analytical forms given above.

Finally, for a general loading case, these uniaxial functions were combined in the fashion

\[
g(a_1, a_3, a_4) = \frac{a_3}{a_3^*} g_3(a_3) + \frac{a_4}{a_4^*} \left[g_4(a_4) + g_1(a_1)\right]
\]

(21)

where \(a_3^*\) and \(a_4^*\) are values of the stress invariants at the current yield surface as determined from the uniaxial load cases \(\sigma_{22} - \epsilon_{22}\) and \(\sigma_{12} - \epsilon_{12}\), respectively.

As originally proposed, the invariant–based plasticity model may exhibit serious errors when the material is subjected to some particular loadings. Consider a biaxial loading example in which \(\sigma_{11} = 5\sigma_{22}\). The properties of the fiber and matrix for this composite are as before. Hence, behavior of the composite under uniaxial stress is as shown in Figure 2. The results from the original invariant–based model compared with micromechanics analysis are shown in Figure 3. From this figure, we notice that the original invariant–based model predicted no yielding at all. Yielding from micromechanics analysis occurs at a transverse tension strain of \(\epsilon_{22} = 0.0003\) whereas the longitudinal stress–strain relation in Figure 2 is linear when strain \(\epsilon_{22}\) is less than 0.0003, which leads to \(g_3(a_3) = 0\). Furthermore, since there is no longitudinal shear stress, \(a_4\) is zero. Thus, the invariant–based model predicts that the material remains elastic.
Modified Invariant–Based Flow Rule

To overcome the problem mentioned above, an alternative scalar hardening coefficient is proposed. Equation (21) defines a moderately simple form for $g(\phi)$. Certainly, the second invariant $a_2$ could be added to permit a separation of transverse tension response from transverse shear. However, for this study, we follow the lead of Hansen et al. [1991] and neglect $a_2$. Consideration of specific load paths in which the invariants $a_3$ and $a_4$ are isolated leads to expressions for $g_3(a_3)$ and $g_4(a_4)$ from Eqs. (19) and (20).

As discussed above, the behavior of the composite material is essentially elastic in the fiber direction, so it is not effective to develop $g_1(a_1)$ based upon a uniaxial stress state of longitudinal tension. However, there is need to represent the effect of longitudinal tension on longitudinal shear yielding. Hence, $g_1(a_1)$ is developed from a somewhat arbitrary biaxial condition in which $\sigma_{11} = 10\sigma_{12}$. Following the development procedure as before, we write

$$
d\epsilon_p = \left[ \frac{2}{3} \frac{(F + 2G)(2G + 0.02M)}{(2G)^2} \right]^{\frac{1}{2}} d\epsilon_{11}^p
$$

$$
d\sigma_p = \left[ \frac{3}{2} \frac{(2G + 0.02M)}{F + 2G} \right]^{\frac{1}{2}} d\sigma_{11}
$$

$$
\frac{1}{H'} = \frac{2}{3} \frac{F + 2G}{2G} \frac{1}{H'_{11}}
$$

Hence, we have

$$
g_1(a_1) = \frac{1}{16a_1^2G(G + 0.01M)} \frac{1}{H'_{11}}
$$

(22)

where

$$
H'_{11} = \frac{d\sigma_{11}}{d\epsilon_{11}^p}
$$
The modified scalar hardening function under multiaxial loads is chosen to be

\[ g(a_1, a_3, a_4) = \frac{a_1}{a_1^*} g_1(a_1) + \frac{a_3}{a_3^*} g_3(a_3) + \frac{a_4}{a_4^*} g_4(a_4) \]  \hspace{1cm} (23)

which reduces to the specific forms defined by Eqs. (19), (20), and (22) for the corresponding load cases. In Eq. (23), \( a_1, a_3, \) and \( a_4 \) are the stress invariants; \( a_1^*, a_3^* \) and \( a_4^* \) have the same meanings as before. From Figure 3, we can see that the results from the modified invariant-based model are substantially better than those from Hansen's original model.

It should be noted that the hardening function proposed by Hansen et al., Eq. (21), was not intended to be generally applicable. It performed reasonably well for the specific material and stress states that were of interest in their studies. The hardening function proposed here, Eq. (23), is simply an alternative which permits yielding to occur in the case cited above. Substantially more research remains to further generalize the expression for the scalar hardening coefficient \( g(\bar{\sigma}) \) and to quantify the coupling between longitudinal tension and longitudinal shear.

**Implementation**

Software for the two material models under consideration in this study was implemented in FINITE, a program for linear, nonlinear, static and dynamic analysis [Dodds & Lopez, 1980]. The implementation is general enough to permit the solution of 2-D (plane stress, plane strain and axisymmetric) and 3-D nonlinear problems for orthotropic materials. The plasticity rate equations for the Hill model were integrated by the tangent stiffness–radial corrector approach, a one-step, forward Euler method; see [Dodds, 1987]. The invariant-based model was implemented with a tangent stiffness approach but the implementation lacked a return technique due to the absence of an effective strain definition from which to establish an appropriate yield surface position. Hence, the implementation is defective in that some spurious hardening will occur under multiaxial loading. However, for the purposes of this study, the spurious hardening is adequately controlled by appropriate
use of strain subincrements [Dodds, 1987]. Uniaxial stress states will be unaffected; radial loading is represented exactly. The following contains a development of the tangent operator \([D_{ep}]\) for each of the material models. Complete details of the implementation are given in [Wang, 1991].

**Hill's Model**

For a transversely isotropic material, the plastic strain increment, in vector notation, is written as

\[
[d\varepsilon^p] = d\lambda \left[ \frac{\partial \phi}{\partial \sigma} \right] = d\lambda [\sigma^\ast] 
\]  

(24)

where

\[
d\lambda = g(\phi)\{\sigma^\ast\}^T\{d\sigma\} 
\]  

(25)

and \(\{\sigma^\ast\} = \left\{ \frac{\partial \phi}{\partial \sigma} \right\} = \begin{bmatrix} \sigma^\ast_{11} & \sigma^\ast_{22} & \sigma^\ast_{33} & \sigma^\ast_{12} & \sigma^\ast_{13} & \sigma^\ast_{23} \end{bmatrix}^T\) has components:

\[
\begin{align*}
\sigma^\ast_{11} &= \frac{\partial \phi}{\partial \sigma_{11}} = 2[G(\sigma_{11} - \sigma_{33}) + G(\sigma_{11} - \sigma_{22})] \\
\sigma^\ast_{22} &= \frac{\partial \phi}{\partial \sigma_{22}} = 2[F(\sigma_{22} - \sigma_{33}) + G(\sigma_{22} - \sigma_{11})] \\
\sigma^\ast_{33} &= \frac{\partial \phi}{\partial \sigma_{33}} = 2[G(\sigma_{33} - \sigma_{11}) + F(\sigma_{33} - \sigma_{22})] \\
\sigma^\ast_{12} &= \frac{\partial \phi}{\partial \sigma_{12}} = 4M\sigma_{12} \\
\sigma^\ast_{13} &= \frac{\partial \phi}{\partial \sigma_{13}} = 4M\sigma_{13} \\
\sigma^\ast_{23} &= \frac{\partial \phi}{\partial \sigma_{23}} = 4(G + 2F)\sigma_{23}
\end{align*}
\]  

(26)

The strain increment \(\{d\varepsilon\}\) is separated into elastic \(\{d\varepsilon^e\}\) and plastic \(\{d\varepsilon^p\}\) parts according to

\[
\{d\varepsilon\} = \{d\varepsilon^e\} + \{d\varepsilon^p\} 
\]  

(27)
The elastic strain increments are related to the stress increments by
\[
\{ \varepsilon^e \} = [S] \{ \sigma \}
\]
(28)
or conversely
\[
\{ \sigma \} = [D]^e \{ \varepsilon^e \}
\]
(29)
where \([S]\) is the elastic compliance matrix, \([D]^e\) is the elastic material stiffness, and \([S] = [D]^e\)^{-1}. In Eq. (29), substitution of for \(\{ \varepsilon^e \}\) from Eq. (27) and then for \(\{ \varepsilon^p \}\) from Eq. (24) gives
\[
\{ \sigma \} = [D]^e \{ \varepsilon \} - d\lambda [D]^e \{ \varepsilon^r \}
\]
(30)
We note that the only unknown parameter in Eq. (30) is the positive scalar \(d\lambda\). In order to derive the elastic–plastic material stiffness, an expression for \(d\lambda\) must first be determined. From Eqs. (1), (2), and (6), the hardening parameter \(\bar{\phi}\) can be related to the effective stress \(\bar{\sigma}_e\) by
\[
\bar{\phi} = \frac{2(F + 2G)}{3} \bar{\sigma}_e^2
\]
(31)
Eq. (1) can be rewritten as follows
\[
\Phi = \phi - \frac{2(F + 2G)}{3} \bar{\sigma}_e^2
\]
(32)
The consistency condition requires that the yield criterion be satisfied as long as the material is in a plastic state. It is expressed by \(d\Phi = 0\). From Eq. (32),
\[
d\phi = \frac{4(F + 2G)}{3} \bar{\sigma}_e d\bar{\sigma}_e
\]
(33)
Equating the increment in plastic work done by the multiaxial stress state to that done by the uniaxial effective stress, we have

\[ dW_p = \bar{\sigma}_e d\bar{\varepsilon}_p = \frac{4(F + 2G)}{3} \bar{\varepsilon}_e d\lambda \]  

(34)

Thus

\[ d\bar{\varepsilon}_p = \frac{4(F + 2G)}{3} \bar{\sigma}_e d\lambda \]

(35)

By applying Eqs. (11), (33), and (35), we eliminate \( d\phi \) from Eq. (33).

\[ d\phi = \frac{4(F + 2G)}{3} \bar{\sigma}_e d\bar{\varepsilon}_e = \frac{4(F + 2G)}{3} \bar{\sigma}_e \frac{d\bar{\varepsilon}_e}{d\bar{\varepsilon}_p} d\bar{\varepsilon}_p \]  

(36a)

\[ d\phi = \frac{4(F + 2G)}{3} \bar{\sigma}_e H' d\bar{\varepsilon}_p \]  

(36b)

\[ d\phi = \frac{4(F + 2G)}{3} \bar{\sigma}_e H' \frac{4(F + 2G)}{3} \bar{\sigma}_e d\lambda = \frac{16(F + 2G)^2}{9} H' \bar{\sigma}_e^2 d\lambda \]  

(36c)

Finally,

\[ [\sigma^*]^T [D_e] [d\varepsilon] - \frac{16(F + 2G)^2}{9} H' \bar{\sigma}_e^2 d\lambda = 0 \]  

(37)

From Eq. (30), the consistency condition, Eq. (37), is rewritten as

\[ [\sigma^*]^T [D_e] [d\varepsilon] - [\sigma^*]^T [D_e] [\sigma^*] d\lambda - \frac{16(F + 2G)^2}{9} H' \bar{\sigma}_e^2 d\lambda = 0 \]  

(38)

Solving for \( d\lambda \), we obtain

\[ d\lambda = \frac{[\sigma^*]^T [D_e] [d\varepsilon]}{\frac{16(F + 2G)^2}{9} H' \bar{\sigma}_e^2 + [\sigma^*]^T [D_e] [\sigma^*]} \]  

(39)
To relate the stress increment $d\sigma$ to the total strain increment $d\varepsilon$, we use Eq. (30) and (39) and obtain the final expression for $[D]_{kp}$

$$[d\sigma] = [D]_{kp}[d\varepsilon]$$

$$[D]_{kp} = [D]_e - \frac{[D]_e[\sigma^*][\sigma^*]^T[D]_e}{\frac{16}{9}(I+2G)^2 H^2\sigma_e^2 + [\sigma^*]^T[D]_e[\sigma^*]}$$

As shown in Wang [1991], the elastic–plastic material stiffness from Hill’s theory reduces to the elastic–plastic material stiffness from the von Mises theory when the material is isotropic.

**Invariant–Based Model**

By regrouping the flow rule, Eq. (5), we have

$$[d\varepsilon^p] = g((\sigma))[\sigma^*]^T[d\sigma][\sigma^*] = g((\sigma))[A][d\sigma]$$

where $[A] = [\sigma^*][\sigma^*]^T$. Substitution of Eqs. (28) and (42) into Eq. (27) produces $[d\varepsilon]$ as

$$[d\varepsilon] = [S][d\sigma] + g((\sigma))[A][d\sigma] \quad \text{or} \quad [d\varepsilon] = [[S] + g((\sigma))[A]][d\sigma]$$

Equation (43) may be written as

$$[d\varepsilon] = [A^*][d\sigma] \quad \text{where} \quad [A^*] = [S] + g((\sigma))[A]$$

To find $[D]_{kp}$ which relates $[d\sigma]$ to $[d\varepsilon]$, we simply invert matrix $[A^*]$,

$$[d\sigma] = [A^*]^{-1}[d\varepsilon]$$

The final form of the elastic–plastic material stiffness can be written as

$$[D]_{kp} = [[S] + g((\sigma))[A]]^{-1}$$
Results

The two material models presented in this work can be applied to a wide range of problems. A few of these applications are demonstrated in this section. First, a variety of uniaxial and biaxial stress–strain plots for an individual material point are developed; the modified invariant and Hill’s plasticity models are used. Emphasis is placed on the comparisons with micromechanics analysis. Finally, both material models are used to predict results for an off–axial tension study for which experimental data exist.

In order to compare the results from the Hill model with the micromechanics analysis, a specific uniaxial load path has to be chosen to determine the scalar hardening function. In this study, two load paths are selected to illustrate the variability in predicted behavior from the model. One is from longitudinal shear data (denoted tau–12 in the figures), and the other is from transverse tension data (denoted sigma–22). Longitudinal tension response of the selected composite is essentially linear elastic. Hence, use of this load path is inappropriate for this comparison.

Figure 4 illustrates a comparison of the Hill and the invariant models with micromechanics analysis for uniaxial longitudinal shear and uniaxial transverse tension. If the effective stress–effective strain relationship for the uniaxial longitudinal shear loading is determined from longitudinal shear data, and the effective stress–effective strain relationship for the transverse tension loading is determined from transverse tension data, results from Hill’s model fall extremely close to the micromechanics prediction. Invariant results are quite precise without initial selection of the appropriate load path. Conversely, Hill’s formulation produces substantial discrepancies when the proper effective stress–effective strain curve is not selected a priori. This figure clearly indicates the results from the Hill formulation are critically dependent on the loading path that is used to define the effective stress–effective strain relation, whereas the invariant model shows no such bias.
Figure 5 represents the $\sigma_{11}-\epsilon_{11}$ relation and the $\sigma_{12}-\gamma_{12}$ relation for a biaxial load case, the loading path is $\sigma_{11} = 10\sigma_{12}$. In Figure 5(a), the behavior of the material from micromechanics analysis is nearly linear. The results from the invariant-based formulation track those from micromechanics analysis, while the two Hill solutions predict a softer response. In Figure 5(b), although neither of these theories agree very well with the micromechanics analysis, the invariant-based model performs better than the Hill model.

Figure 6(a) presents a $\sigma_{11}-\epsilon_{11}$ stress-strain relationship for a biaxial loading where $\sigma_{11} = 5\sigma_{22}$. The invariant-based theory again follows the micromechanics prediction while the Hill theory is softer. Figure 6(b) shows the $\sigma_{22}-\epsilon_{22}$ stress-strain relationship for the same loading case. As before, Hill’s formulation is softer, while the invariant-based formulation best approaches the micromechanics results.

For combined transverse tension and longitudinal shear, where $\sigma_{22} = 2\sigma_{12}$, the stress-strain relations of $\sigma_{22}-\epsilon_{22}$ and $\sigma_{12}-\gamma_{12}$ are shown in Figure 7. Neither the invariant-based formulation nor Hill’s formulation match the micromechanics prediction very well. However, the invariant-based model yields better results than the Hill model.

So far, all of loadings are assumed to be proportional and monotonically increasing; the last problem deals with non-proportional loading. Transverse tension stress $\sigma_{22}$ is applied up to the yield stress $\sigma_{22}^{\text{y}}$. Then longitudinal shear $\sigma_{12}$ is applied, while maintaining $\sigma_{22} = \sigma_{22}^{\text{y}}$. For this biaxial loading, the loading path and the resulting longitudinal shear response is shown in Figure 8. If the effective stress–effective strain relationship is determined from longitudinal shear data, both the Hill model and the invariant-based model agree reasonably well with the micromechanics results. However if transverse tension is selected as the dominant stress state, predictions from Hill’s model contain substantial error.
From these comparisons of a single material point, one can see, first, although we have different results for Hill’s theory when different effective stress–effective strain relations are used, the discrepancy is not always large. In fact, uniaxial loading appears to be the most challenging test of Hill’s model. Second, the invariant–based formulation usually represents a measurable improvement when compared to Hill’s formulation.

Finally, analysis of an off–axis tensile specimen, composed of boron–epoxy laminate, is considered. The results are compared between the invariant–based model and Hill’s model in which the scalar hardening coefficient is selected from two different uniaxial tests. The specimen is of the type shown in Figure 9 \((L/h = 20)\), fibers are aligned at \(30^\circ\) relative to the \(x\) direction. The directional material properties are shown in Figure 10.

Eight–node isoparametric elements are selected to idealize the specimen. Boundary conditions are such that the ends \((x = 0, L)\) remain plane and parallel. Full integration is employed. A plane stress analysis is conducted.

Figure 11 shows predicted results from both models; the experimental results of Cole and Pipe [1975] are also shown. This test again confirms that the invariant–based formulation may yield better results than Hill’s formulation.

**Conclusion**

For best performance, Hill’s model requires *a priori* knowledge of the dominate stress component so that the appropriate stress–strain relation can be chosen as the effective stress–effective strain relation. This is overly restrictive for some composite materials due to the assumption that a unique effective stress–effective strain relation exists.

The invariant–based theory, proposed by Hansen *et al.* requires no assumption of an effective stress–effective strain relation, and is able to model plastic behavior of transversely materials over a full range of loading possibilities. However, it may still exhibit errors for some stress states. Thus, a modified form of the scalar hardening function has been proposed. Through this study, the invariant–based theory proved to model transversely isotropic
materials with the same or greater accuracy than Hill's theory. Further extension of the invariant-based theory to include the invariant $a_2$ in the hardening function would add additional generality to the approach and permit modeling of transverse shear response.

Appendix I – References


**Appendix II – Notation**

$a_1, a_2, a_3, a_4, a_5$ Stress invariants.

$dW_p$ Plastic work increment.

$d\lambda$ Scalar proportionality factor.

$[D]_e, [D]_{kp}, [S]$ Elastic material stiffness, elastic-plastic material stiffness, elastic compliance matrix.

$E, E_{11}, E_{22}, E_T$ Modulus of elasticity, tangent modulus.

$F, G, M$ Characteristic parameters of anisotropy.

$G, G_{12}, G_{13}, G_{23}$ Shear modulus.

$g, g(\phi), g(\bar{\phi})$ Scalar hardening parameter.

$H', H'_{11}, H'_{22}, H'_{12}$ Slope of effective stress–effective plastic strain curve.

$X, Y, S_{12}$ Uniaxial yield strengths.

$\Phi, \phi, \bar{\Phi}$ Yield function quantities.

$\nu, \nu_{12}, \nu_{13}, \nu_{23}$ Poisson’s ratio.

$\sigma_{11}, \sigma_{22}, \sigma_{12}$ Components of stress with respects to orthotropic material coordinates.

$\sigma_c, d\epsilon_p$ Effective stress, effective plastic strain

$\sigma_{ij}, \epsilon_{ij}$ Stress and strain tensors.
Figure Captions

Figure 1. Finite Element Mesh Used for Micromechanics Analysis

Figure 2. Uniaxial Stress–Strain Relations for Example Composite

Figure 3. Performance of Original and Modified Forms of the Invariant–Based Model Under Biaxial Loading: \( \sigma_{11} = 5\sigma_{22} \)

Figure 4. Performance of the Invariant and Hill’s Models, Uniaxial Loading
   (a) Longitudinal Shear Loading
   (b) Transverse Tension Loading

Figure 5. Performance of the Invariant and Hill’s Models, Biaxial Loading: \( \sigma_{11} = 10\sigma_{12} \)
   (a) Longitudinal Tension Response
   (b) Longitudinal Shear Response

Figure 6. Performance of the Invariant and Hill’s Models, Biaxial Loading: \( \sigma_{11} = 5\sigma_{22} \)
   (a) Longitudinal Tension Response
   (b) Transverse Tension Response

Figure 7. Performance of the Invariant and Hill’s Models, Biaxial Loading: \( \sigma_{22} = 2\sigma_{12} \)
   (a) Transverse Tension Response
   (b) Longitudinal Shear Response

Figure 8. Performance of the Invariant and Hill’s Models, Nonproportional Loading.
   (a) Loading Path
   (b) Longitudinal Shear Response

Figure 9. Off–Axis Tensile Specimen

Figure 10. Uniaxial Stress–Strain Relations for Boron–Epoxy Narmco 5505

Figure 11. Off–Axis Tension of Boron–Epoxy, Fiber Angle = 30°
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Figure 2 – Uniaxial Stress–Strain Relations for the Example Composite
Figure 3 – Performance of Original and Modified Forms of the Invariant-Based Model Under Biaxial Loading: $\sigma_{11} = 5\sigma_{22}$
Figure 4 - Performance of the Invariant and Hill Models, Uniaxial Loading
(a) Longitudinal Shear Loading
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Figure 5 - Performance of the Invariant and Hill Models, Biaxial Loading: $\sigma_{11} = 10\sigma_{12}$
(a) Longitudinal Tension Response, (b) Longitudinal Shear Response
Figure 6 - Performance of the Invariant and Hill Models, Biaxial Loading: \( \sigma_{11} = 5\sigma_{22} \)

(a) Longitudinal Tension Response

(b) Transverse Tension Response
Figure 7 - Performance of the Invariant and Hill Models, Biaxial Loading: $\sigma_{22} = 2\sigma_{12}$
(a) Transverse Tension Response,  
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\[ E_{11} = 207.69 \text{ GPa} \]
\[ E_{22} = 19.8 \text{ GPa} \]
\[ G_{12} = 6.07 \text{ GPa} \]
\[ \nu_{12} = 0.225 \]
Figure 11 – Off-Axis Tension of Boron-Epoxy, Fiber Angle = $30^\circ$